Power-law Persistence characterizes Travelling Waves in Coupled Circle Maps with Repulsive Coupling

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We study persistence in coupled circle maps with repulsive (inhibitory) coupling, and find that it offers an effective way to characterize the synchronous, travelling wave and spatiotemporally chaotic states of the system. In the travelling wave state, persistence decays as a power law and, in contrast to earlier observations in dynamical systems, this power-law scaling does not occur at the transition point alone, but over the entire dynamical phase (with the same exponent). We give a cellular automata model displaying the qualitative features of the travelling wave regime, and provide an argument based on the theory of Motzkin numbers in combinatorics to explain the observed scaling.

PACS numbers: 05.45.RA,05.70.Fh,02.50.-r

I. INTRODUCTION

The study of extended dynamical systems is relevant to the understanding of spatiotemporal phenomena observed in diverse fields, ranging from condensed matter physics to biology. One of the well-established models for such complex systems is the coupled map latice (CML), which captures several features of pattern formation in these systems [1]. Such extended systems are also test-beds for finding valuable links between statistical mechanical formalisms and dynamical systems theory. The work here relates travelling waves, a phenomenon observed in a range of extended systems, to persistence, a concept widely researched in the context of nonequilibrium statistical mechanics.

The phenomenon of travelling waves is of interest in several contexts. For instance, in biology, numerous processes are described by finite chains of coupled nonlinear oscillators and travelling waves of activity in such systems can be seen in processes such as undulatory motion of swimming organisms such as lamprey, leech and xenopus tadpole, waves of excitation arising in sensory processing in the cortex and the rhythmic contraction of smooth muscles to propel food in the digestive tract [2].

Specifically, in this paper, we study the travelling regime in coupled phase oscillators with *repulsive coupling*. In biological systems, the presence and importance of inhibitory connections is well acknowledged. Couplings between neurons are both excitatory and inhibitory [3]. Ecological webs typically have both positive and negative connections between their components [4]. In certain laboratory systems like coupled lasers, negative couplings have been investigated [5]. Laplacians with negative coefficients arise in several equations in pattern formation. The well-known Swift-Hohenberg and Kuramoto-Sivashinsky equations have such a term.

One of the core issues of basic theoretical interest here is the following: *Are transitions in deterministic chaotic systems governed by the universality classes of known stochastic or equilibrium models* [7, 8]? In the context of spatially extended systems, several studies on dynamic phase transitions are triggered by this important question. The dynamic transitions studied so far have mainly been in the context of synchronization [9]. All transitions to an absorbing state (namely to an inactive state, with no fluctuations) have been conjectured to belong to the universality class of directed percolation. Since synchronized state is an absorbing state, several authors have attempted to verify this conjecture in the context of transition to synchronized state [10–12].

In this paper we focus on the transition to the travelling wave pattern in coupled circle maps with inhibitory coupling. We find the concept of *persistence*, that has hitherto been used mainly in the context of non-equilibrium models, is a very useful quantifier to characterize the travelling wave regime. This is significant in view of the fact that *no order parameter has yet been proposed to capture the transition to travelling wave regime* in this class of models.

Persistence in the context of stochastic processes is defined as the probability $P(t)$ that a stochastically fluctuating variable has not crossed a threshold value up to time t [13]. For a spin system with discrete states, such as Ising or Potts spins, it is defined in terms of the probability that a given spin has not flipped out of its initial state up to time t. The power-law tail of $P(t)$, *i.e.*

$$P(t) \sim 1/t^{\theta}, \tag{1}$$

defines the persistence exponent θ. The probability $P(t)$ is averaged over an ensemble of random initial conditions.

Of late, persistence exponents for stochastic systems have attracted attention as they seem to be a new class of exponents, not derivable in general from the other static and dynamic exponents [13]. It is important to note that the persistence exponent is known to be far less universal than other three exponents that typically characterize scaling behaviour of quantities of physical interest: η, ν and the dynamical exponent z [10]. The reason is that persistence probes the full, in general *non-Markovian*, time evolution of a local fluctuating variable, such as a spin or density field, from its initial state [15]. Thus, universality of the persistence exponents, in particular for systems with deterministic chaos, is *a priori* unexpected.

Some studies on persistence that have recently been carried out in the context of synchronization in CMLs support the idea that it is in the class of directed percolation [11]. However, in spatially extended systems one typically obtains more complex dynamical behaviour than synchronization. Here, we investigate transitions from the synchronous state to the travelling wave state, and then into spatiotemporal chaos in a CML. Interestingly, we find that the persistence offers an effective way to characterize these dynamical phases and the transition between them. It is surprising that a single quantifier works as a good indicator of different complex phases.

II. MODEL

We focus on coupled circle maps with negative coupling. Despite its presence in several systems, negative coupling in CMLs has not been studied except in a work by Neufeld and Vicsek [16], though there have been some recent works on inhibitory coupling in coupled phase oscillators [17, 18].

We will first introduce the model and present the salient features of the transitions to the travelling wave regime in this system. We will then demonstrate the efficacy of persistence as an order parameter to characterize this transition. Further, we will introduce a cellular automata model yielding the qualitative and quantitative features of the travelling wave's dynamical phase.

We assign a continuous variable $x_i(t)$ at each site i at time t where $1 \le\ i \le N$. The evolution of $x_i(t)$ is defined by

$$x_i(t+1) = f(x_i(t)) \ - \ \frac{\epsilon}{2}\,(x_{i-1}(t) + x_{i+1}(t) - 2\,x_i(t)) \tag{2}$$

The parameter ϵ is the coupling strength and the function $f(x)$ is the circle map, $f(x) = x + \omega - \frac{k}{2\pi}\, sin(2\pi x)$. We confine the dynamics to the interval $[0, 1]$ using the following rule. If $int[x_i(t)] = m$, $x_i(t) = x_i(t) - m$ if $x_i(t) > 0$ and $x_i(t) = x_i(t) - m + 1$ if $x_i(t) < 0$.

The fixed point solution for the local map $f(x)$ is given by $x^* = \frac{1}{2\pi}\sin^{-1}(\frac{2\pi\omega}{k})$. The system allows a synchronous fixed point such that if $x_i(t) = x^*$ for $1 \le i \le N$ at some time t, the variable value $x_i(t') = x^*$ for any site i for all $t' \ge t$. Linear stability analysis for the synchronous fixed point can be carried out. The condition for a synchronous fixed point is a necessary and not sufficient condition for observing synchronization [19]. However, the fixed point acts as an excellent reference point as in the case of symbolic dynamics, and we can define the dynamics at site i to be laminar if $|x_i(t) - x^*| < \delta$ for a small enough δ and to be turbulent otherwise.

III. RESULTS

We first describe the qualitative behavior of the above model (Eq. 1). In Fig. 1, we show space-time density plots of laminar regions for three different choices of ϵ. The turbulent regions are dark in this representation. We fix $\omega = 0.068$. We clearly see that for $\epsilon = 0.38$, the system shows spatiotemporal chaos. For $\epsilon = 0.3$, we observe annihilation of left-moving and right-moving disturbances; an imbalance of disturbances on the left and right will eventually lead to the development of a traveling wave on one of the two sides. This pattern is observed for a wide parameter range between $0.155 \le \epsilon \le 0.36$. For smaller values of ϵ, say $\epsilon = 0.14$, we observe a synchronized state and there are no turbulent sites after a transient.

The space-time evolution of circle maps with inhibitory coupling is very similar to that of the system with positive coupling [10, 11], but with an interesting change: for positive coupling the disturbance spreads out in both directions (left and right) and then disappears, while in our case (Fig. 1b) a disturbance travels along one direction (left or right). When it meets a disturbance travelling in the opposite direction, they annihilate each other (seen as a triangle in the figure) [20].

We define the local persistence $P(t)$ as the probability that a local state variable $x_i(t)$ does not cross the fixed point value x^* up to the time t [11]. Let $P_+(t)$ be the fraction of sites for which $x_i(0) > x^*$, and $x_i(\tau) > x^*$ for $1 \le \tau \le t$, and $P_-(t)$ be the fraction of sites for which $x_i(0) < x^*$, and $x_i(\tau) < x^*$ for $1 \le \tau \le t$. Obviously, persistence $P(t) = P_+(t) + P_-(t)$.

In Fig. 2, we show $P(t)$ as a function of time for various ϵ values. We find that it is an excellent indicator of the dynamics. $P(t)$ decays almost exponentially for the spatiotemporally chaotic state. For the synchronous state, $P(t)$ saturates. In the entire travelling wave state, $P(t)$ decays as a power law with the *same exponent.* A clear asymptotic $1/t$ decay is shown for three very different ϵ.

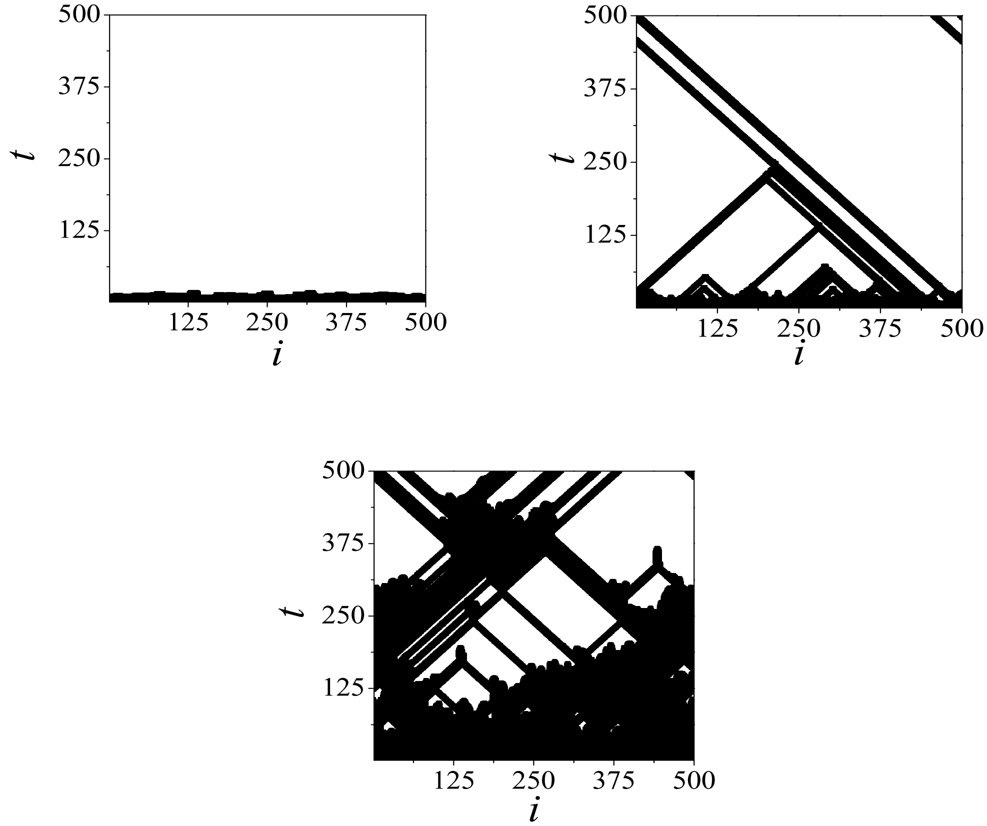
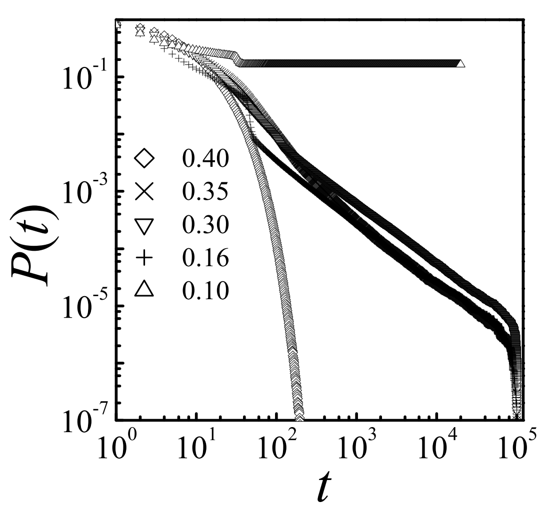


FIG. 1: Space-time plot of CML for $\omega = 0.068$ and $N = 500$ for three different ϵ values. Turbulent sites are dark in this representation. (a) $\epsilon = 0.14$, where the turbulent spots vanish, implying a synchronous state (b) $\epsilon = 0.3$, where we clearly see the turbulent spots moving like travelling waves; this feature is seen for $0.155 \leq \epsilon \leq 0.36$. (c) $\epsilon = 0.38$, where the system shows spatiotemporal chaos.

FIG. 2: a) $P(t)$ as a function of t for CML with $\omega = 0.068$, $N = 10^5$. for $\epsilon = 0.1, 0.16, 0.3, 0.35$ and 0.4. (from top to bottom). We average over at least 400 initial conditions. Error bars are of the size of the symbol.



$L\ L\ L$	$L\ L\ R$	$L\ R\ L$	$L\ R\ R$	$R\ L\ L$	$R\ L\ R$	$R\ R\ L$	$R\ R\ R$	$L\ L\ 0$	$L\ R\ 0$
L	0	0	0	L	0	R	R	0	0
$L\ 0\ L$	$L\ 0\ R$	$0\ L\ L$	$0\ L\ R$	$0\ R\ L$	$0\ R\ R$	$R\ 0\ L$	$R\ 0\ R$	$R\ R\ 0$	$R\ L\ 0$
L	0	L	0	0	0	0	R	R	0
$L\ 0\ 0$	$0\ L\ 0$	$0\ 0\ L$	$R\ 0\ 0$	$0\ R\ 0$	$0\ 0\ R$	$0\ 0\ 0$			
0	0	L	R	0	0	0			

TABLE I: The value of a site at the next time instant is given as a function of itself and nearest neighbors for the CA model.

Our principal result is the following: the local persistence exponent $\theta_l = 1$ in the travelling wave state. A significant difference with power-law scaling observed earlier in CMLs [11] is the following: Generically one obtains a power law decay of the order parameter at the *transition point alone* for a continuous transition. What we observe here is very different. Power-law scaling characterizes the entire dynamical phase, namely the *presence of power-law scaling of the persistence is a property of the travelling wave regime.*

If we look at the space-time plots, we observe that disturbances moving along separate directions merge. Let us call the right-moving disturbances R and left-moving disturbances L. The annihilation can be described by $R + L = 0$. No defects seem to emerge spontaneously. Thus if the number of Rs is more than Ls, we will see an overall right-moving wave emerging asymptotically. This is a very interesting case of broken symmetry, driven by fluctuations. The equations are perfectly isotropic. However, the disturbances at each point pick up a direction and move along it. The description $R+L = 0$ would indicate an approximate conservation law here. Namely, the absolute difference between the number of Rs and Ls is conserved. On the other hand, in the spatiotemporally chaotic state, the disturbances can spontaneously emerge and there is no evidence of a conservation law.

IV. CELLULAR AUTOMATA MODEL

Now we attempt to obtain this power-law behavior from the above qualitative description of travelling waves using a 3-state cellular automata (CA) model. We propose a one dimensional model where each site can take 3 values R, L and 0. R is a particle moving to the right. L is a particle moving to the left and 0 represnts the absence of the particle. The evolution rule is simple. R at site $i-1$ moves to the site i unless there is a L at site i or $i+1$. If site i and $i+1$ both have L, we have L at site i at the next instant. If both i and $i-1$ have R, site i takes value R at the next instant. If there is an R at site $i-1$ and if one of the sites i and $i+1$ is L and other is 0, site i becomes 0. Similarly, L at site $i+1$ moves to the left, *i.e.* to site i, unless there is an R at site i or site $i-1$. In all other cases, site i takes value 0 at next time instant.

The rules are given explicitly in Table 1. Traveling waves emerge spontaneously for an imbalance of R and L in the initial conditions. We show the space-time plot of the CA model in Fig.3a and it is strikingly similar to Fig.1b. The persistence in this model is plotted in Fig.3b and it clearly decays as $1/t$.

For analysing this behavior, let us look at individual values. If the initial value was L or R, it is unlikely to persist. R at site i will persist for t time steps if there is a block of t consecutive sites from $i-t+1$ to site i which are R and there is no L for sites $i+1$ to $i+t-1$. The probability of t consecutive R or L sites decays exponentially with t for a random initial configuration giving negligible contribution. Persistence at long times turns out to be determined by the persistence of zeros from the analysis below.

A zero at site i persists as long as it does not encounter a R or L. If we ignore the right side, R at site $i-t$ will arrive at site i at time t unless cancelled by L between $i-t$ and i. If no other R crosses site i till time t, i will persist till time t. If we assign values $S(R) = 1$, $S(L) = -1$ and $S(0) = 0$, the zero at site i will certainly persist till time t if $\Sigma_{j=i-m}^{j=m} S(j) < 1$ and $\Sigma_{j=i+m}^{j=m} S(j) > -1$ $\forall m \leq t$. This could be mapped on a first passage time probability of a random walker who could take three steps 0, 1 and -1.

We count the number of paths of length n of such a random walker whose cumulative displacement is 0 at $t = 0$ and $t = n$, and is non-positive for $t < n$. Such paths are known as Motzkin paths and the number of such paths of length n is given by $M_n = \sum_{r=1}^{[n/2]} \frac{n!}{(n-2r)!r!r+1!}$[21, 22]. These numbers are also known as Motzkin numbers. Their asymptotic expression is given by [22]

$$M_n = C3^n n^{-3/2}(1 + O(1/n)) \tag{3}$$

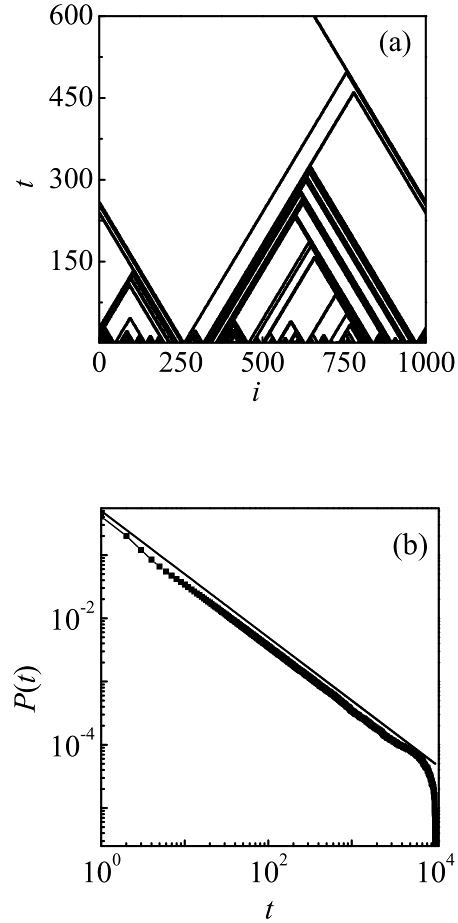


FIG. 3: (a) Representative spatiotemporal plot of the CA model. Sites with nonzero values are dark in this representation. (b) $P(t)$ as a function of time t for the CA model, for $N = 10^5$. We average over 100 initial conditions. Clearly, $P(t)$ goes as t^{-1}. A line $1/2t$ is also plotted for reference.

where $C = \sqrt{27/(4\pi)}$. The probability for first passage to 0 at the $(n-1)$th step and for taking a value 1 at nth step is $f(n) = M_{n-1}\, 3^{-n} \sim n^{-3/2}$. This is similar to a Bernoulli walk. It follows that survival probability $s(n)$ after n steps will go as $n^{-1/2}$ for large n.

Now we compute the persistence probability for 0's. It can occur in several ways. Let $P_0(t)$ be the probability (mentioned above) that the walker on the right does not sum to 1 and one on the left does not sum to -1 till time t. $P_1(t)$ is the probability that the walkers on left and right *simultaneously* sum to -1 and 1 once *at the time* $i_1 \leq t$ and do not cross -1 and 1 till from time i_1 to time t. $P_k(t)$ is the probability that the walkers on left and right *simultaneously* sum to -1 and 1 k times $i_1 < i_2 < i_3 \ldots < i_k \leq t$ and do not cross -1 and 1 from time i_k to t. Clearly $P(t) = \Sigma_{i=0}^{t} P_i(t)$. $P_0(t) = s(t)^2 \propto 1/t$. $P_1(t) = f(t)^2 + \Sigma_{i=1}^{t-1} f(i)^2 s(t-i)^2$. While first term in this expression is $f(t)^2 \sim 1/t^3$, second term is proportional to $\Sigma_{i=1}^{t-1} \frac{1}{i^3(t-i)}$. Using partial fractions, this sum could be shown to be proportional to $\log(t)/t^3$ at the most. Since $P_j(t) < P_1(t)$ for $j > 1$, the correction to $P_0(t)$ from $\Sigma_{i=1}^{t} P_i(t) > tP_1(t) \propto \log(t)/t^2$. Hence the correction to the leading $1/t$ behavior is of the order $\log(t)/t^2$ at the most. Thus we can say that $P(t) \propto 1/t$ for large t [23].

V. SUMMARY

In summary, we have studied coupled circle maps with inhibitory coupling, and observe synchronous, travelling wave and spatiotemporally chaotic states. We find that persistence shows qualitatively different behavior in these

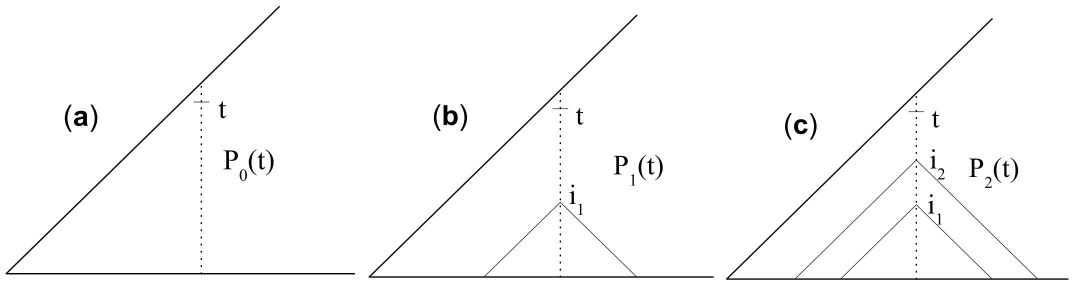


FIG. 4: Schematic representation of events leading to the probablities $P_0(t)$, $P_1(t)$ and $P_2(t)$ are shown in a) b) and c).

different states, offering an effective way to characterize these dynamical phases and the transitions between them.

In particular, we find that persistence for a travelling wave state decays as a power law with the same exponent ($\theta_l = 1$) over the entire dynamical phase. Thus, local persistence provides an excellent quantifier that can act as an order parameter for that phase. Further, we give a 3-state cellular automata model that generates the qualitative features of the travelling wave regime, and provide an argument based on the theory of random walks explaining the observed power-law scaling of the persistence. This study offers an interesting connection between a widely observed dynamical phenomenon and a concept popular in stochastic processes.

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[20] An intuitive argument could be given as follows. Consider equation $\frac{d\psi}{dt} = \epsilon\nabla^2\psi$. Here, changing the sign of ϵ is equivalent to changing the sign of t. Thus negative coupling is somewhat equivalent to evolving back in time. Hence the plots for negative ϵ should be similar to those for positive ϵ, but reversed in time (namely what appears as triangles in the space-time plot of one, appears as inverted triangles in the space-time plot of the other).
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